Introduction

This topic covers how to:

- convert between exponents and logarithms;
- evaluate logarithms manually;
- evaluate logarithms with base 10 or base e on a scientific calculator; and
- apply five logarithm laws to rearrange and simplify logarithms.

Welcome to Numeracy Basics.

This topic will look at Logarithms. Specifically, it will look at how to:

- convert between exponents and logarithms;
- evaluate logarithms manually;
- evaluate logarithms with base 10 or base e on a scientific calculator; and
- apply five logarithm laws to rearrange and simplify logarithms.

Logarithms

Logarithms are the ‘opposite’ of exponents, in the same way as addition and subtraction are opposites and multiplication and division are opposites; that is, the two operations undo each other.

Hence if you have an equation that sets a positive number $a$ equal to an exponent with positive base $b$ and some integer power $c$, it can be rearranged to an equation that sets $c$ equal to the logarithm of $a$ with base $b$:

$$a = b^c, \text{ then } c = \log_b(a)$$
This relationship means that logarithms can be used to solve for an unknown exponent in an exponential equation— which is one of the reasons you may need to use them.

While you simply need to be able to remember this relationship at this stage, by the completion of this topic you should be able to understand how to convert from one equation to the other using the logarithm laws.

In a previous topic we looked at exponents, otherwise known as indices or powers, which are used as a shorthand notation for repeated multiplication. In this topic we focus on logarithms, which are the inverse of exponents. In other words, they are the opposite of exponents in the same way as addition and subtraction are opposites and multiplication and division are opposites— that is, the two operations undo each other.

So if you have an equation that sets a positive number \( a \) equal to an exponent with positive base \( b \) and some integer power \( c \), it can be rearranged to an equation that sets \( c \) equal to the logarithm of \( a \) with base \( b \). You should note here that the \( b \) in the logarithm shown is referred to as the base of the logarithm, in the same way as it is the base of the exponent \( b \) to the power of \( c \).

This relationship means that logarithms can be used to solve for an unknown exponent in an exponential equation— which is one of the reasons you may need to use them. While you simply need to be able to remember this relationship at this stage, by the completion of this topic you should be able to understand how to convert from one equation to the other using the logarithm laws. Before we get to these though, we will look at how to evaluate logarithms.

**Evaluating Logarithms Manually**

The logarithm of a number with a certain base tells you how many times you need to multiply the base by itself in order to obtain the number in question. For example, evaluating \( \log_b a \) tells you how many times you need to multiply \( b \) by itself in order to obtain \( a \). So just how do you evaluate this?

For most bases, evaluating the logarithm simply requires you to perform repeated multiplication of the base until the correct answer is obtained. For example, to evaluate \( \log_2 16 \) you would determine how many times 2 needs to be multiplied by itself in order to equal 16. Repeated multiplication gives a solution of 4, since:

\[
2 \times 2 \times 2 \times 2 = 2^4 = 16
\]

Therefore we have:

\[
\log_2(16) = 4
\]

This tells us that the logarithm of 16 with base 2 is 4, and you should observe that the relationship between the logarithm and exponent are as described previously.
The logarithm of a number with a certain base tells you how many times you need to multiply the base by itself in order to obtain the number in question. For example, evaluating the logarithm of $a$ with base $b$ tells you how many times you need to multiply $b$ by itself in order to obtain $a$. So just how do you evaluate this?

For most bases, evaluating the logarithm simply requires you to perform repeated multiplication of the base until the correct answer is obtained. For example, to evaluate the logarithm of 16 with base 2 you would determine how many times 2 needs to be multiplied by itself in order to equal 16. Repeated multiplication gives a solution of 4, since 2 times 2 times 2 times 2, which we can write as $2^4$, is equal to 16.

Therefore the logarithm of 16 with base 2 is equal to 4, and you should observe that the relationship between the logarithm and exponent are as described previously.

We will discuss how to obtain logarithms of numbers with two particular bases using a scientific calculator later in this topic, but for now we will practice determining logarithms ‘by hand’.

**Examples: Evaluating Logarithms Manually**

Evaluate the following logarithms:

1. $\log_5(125)$
   
   $5 \times 5 \times 5 = 5^3 = 125$
   
   So $\log_5(125) = 3$

2. $\log_4(1024)$
   
   $4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$
   
   So $\log_4(1024) = 5$

3. Rearrange $\log_3(x) = 5$ to give an equation for $x$ in terms of an exponent, and hence solve for $x$

   Rearranging gives $x = 3^5$
   
   Hence $x = 243$

Let’s work through some more examples of evaluating logarithms.

The first example requires us to evaluate the logarithm of 125 with base 5. To do this we determine how many times 5 needs to be multiplied by itself to make 125, which turns out to be three times. So the logarithm of 125 with base 5 is equal to 3.

The second example requires us to evaluate the logarithm of 1024 with base 4. To do this we determine how many times 4 needs to be multiplied by itself to make 1024, which turns out to be five times. So the logarithm of 1024 with base 4 is equal to 5.
The third example requires us to rearrange an equation that sets the logarithm of x with base 3 equal to 5 to an equation that sets x equal to some exponent, and therefore also to solve for x. To do this use the relationship between exponentials and logarithms discussed previously in this topic, to give an equation that sets x equal to three to the power of 5. Therefore x is equal to 243.

Activity 1: Practice Questions

Click on the Activity 1 link in the right-hand part of this screen.

Now have a go at evaluating logarithms on your own by completing some practice questions.

Evaluating Logarithms with Base 10

There are two bases that are used most commonly with logarithms, and there are buttons on a scientific calculator for evaluating these. The first of these bases is 10, and in fact if you see a logarithm written without any base at all you can usually assume the base to be 10. For example:

\[ \log(100\ 000) = \log_{10}(100\ 000) \]

The button for evaluating a log with base 10 on a scientific calculator is the ‘log’ button, so to evaluate the logarithm above you would enter 100 000 and then press ‘log’, giving a result of 5:

\[ \log(100\ 000) = 5 \]

There are two bases that are used most commonly with logarithms, and there are buttons on a scientific calculator for evaluating these. The first of these bases is 10, and in fact if you see a logarithm written without any base at all you can usually assume the base to be 10.

For example, a logarithm written as ‘log one hundred thousand’ refers to the logarithm of one hundred thousand with base 10.
The button for evaluating a log with base 10 on a scientific calculator is the ‘log’ button, so to evaluate the logarithm above you would enter one hundred thousand and then press ‘log’, giving a result of 5. If you like, you can check that multiplying 10 by itself 5 times does indeed give one hundred thousand.

Evaluating Logarithms with Base $e$

The other commonly used base for logarithms is a special number known as Euler’s number, which is denoted by the letter $e$. $e$ is an irrational number that is very important in mathematics, economics and finance, and its first 20 digits are:

$$2.7182818284590452353...$$

A logarithm with base $e$ is referred to as a natural logarithm, and is generally denoted by $\ln$. For example:

$$\ln(7.389) = \log_e(7.389)$$

The button for evaluating a log with base $e$ on a scientific calculator is the ‘ln’ button, so to evaluate the logarithm above you would enter 7.389 and then press ‘ln’, giving a result of approximately 2:

$$\ln(7.389) \approx 2$$
Examples: Evaluating Logarithms with Base 10 or $e$

Evaluate the following logarithms, rounding to two decimal places as necessary:

1. $\log(500)$
   Evaluating using a calculator gives 2.70

2. $\ln(200\,000)$
   Evaluating using a calculator gives 12.21

3. Rearrange $400 = e^x$ to give an equation for $x$ in terms of a natural logarithm, and hence solve for $x$
   Rearranging gives $x = \log_e(400) = \ln(400)$
   Hence $x = 5.99$

Let’s work through some more examples of evaluating logarithms with base 10 or $e$.

The first example requires us to evaluate the logarithm of 500 with base 10. To do this we can enter 500 in a scientific calculator, and then press the ‘log’ button, which gives a result of approximately 2.70.

The second example requires us to evaluate the logarithm of 200 000 with base $e$. To do this we can enter 200 000 in a scientific calculator, and then press the ‘ln’ button, which gives a result of approximately 12.21.

The third example requires us to rearrange an equation that sets 400 equal to $e$ to the power of $x$, to an equation that sets $x$ equal to some natural logarithm, and therefore also to solve for $x$. To do this use the relationship between exponentials and logarithms discussed previously in this topic, to give an equation that sets $x$ equal to the natural logarithm of 400. Therefore $x$ is approximately equal to 5.99.

Activity 2: Practice Questions

Click on the Activity 2 link in the right-hand part of this screen.

Now have a go at evaluating logarithms with base 10 or $e$ on your own by completing some practice questions.
Logarithm Laws 1 and 2

As with exponents, there are some ‘laws’ you need to abide by when working with logarithms. We will cover the five most important ones here.

When detailing these laws, we will use the variables $a$, $b$ and $c$ to represent any positive numbers.

Logarithm Laws 1 and 2 are the simplest two logarithm laws, and are as follows:

**Law 1:** $\log_b(b) = 1$

For example, $\log_{10}(10) = 1$

**Law 2:** $\log_b(1) = 0$

For example, $\log_{10}(1) = 0$

Logarithm Laws 3 and 4

Logarithm Laws 3 and 4 cover how to deal with logarithms of products and divisions respectively, and are as follows:

**Law 3:** $\log_b(ac) = \log_b(a) + \log_b(c)$

For example, $\log_2(4 \times 8) = \log_2(4) + \log_2(8)$

$= 2 + 3$

$= 5$

$= \log_2(32)$

**Law 4:** $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$

For example, $\log_2\left(\frac{64}{4}\right) = \log_2(64) - \log_2(4)$

$= 6 - 2$

$= 4$

$= \log_2(16)$
Logarithm Laws 3 and 4 cover how to deal with logarithms of products and divisions respectively.

Law 3 states that the logarithm of $a$ times $c$ with base $b$ is equal to the logarithm of $a$ with base $b$ plus the logarithm of $c$ with base $b$. In other words, this law states that the logarithm of a multiplication is equal to the sum of the logarithms. To see that this is the case, suppose we have the logarithm of 4 times 8 with base 2. Our law tells us that this is equal to the logarithm of 4 with base 2 plus the logarithm of 8 with base 2 and if we evaluate these logarithms we can determine that they are equal to 2 and 3 respectively. Hence the sum of our two logarithms is 5, which is equal to the logarithm of 32 with base 2 as required.

Law 4 states that the logarithm of $a$ divided by $c$ with base $b$ is equal to the logarithm of $a$ with base $b$ minus the logarithm of $c$ with base $b$. In other words, this law states that the logarithm of a division is equal to the difference of the logarithms. To see that this is the case, suppose we have the logarithm of 64 divided by 4 with base 2. Our law tells us that this is equal to the logarithm of 64 with base 2 minus the logarithm of 4 with base 2 and if we evaluate these logarithms we can determine that they are equal to 6 and 2 respectively. Hence the difference of our two logarithms is 4, which is equal to the logarithm of 16 with base 2 as required.

Logarithm Law 5

Logarithm Law 5 covers logarithms of exponents, and is as follows:

**Law 5:** $\log_b(a^c) = c \log_b(a)$

For example, $\log_2(8^2) = 2 \log_2(8)$

\[
= 2(3) = 6 = \log_2(64)
\]

Logarithm Law 5 covers logarithms of exponents.

It states that the logarithm of $a$ to the power of $c$ with base $b$ is equal to $c$ times the logarithm of $a$ with base $b$. To see that this is the case, suppose we have the logarithm of 8 to the power of 2 with base 2. Our law tells us that this is equal to two times the logarithm of 8 with base 2, and if we evaluate this logarithm we can determine that it is equal to 3. Hence the product of our logarithm and 2 is 6, which is equal to the logarithm of 64 with base 2 as required.
Examples: Logarithm Laws

1. Evaluate $\log_{15}(15) + \log_3(3)$
   
   $\log_{15}(15) + \log_3(3) = 1 + 1$
   
   $= 2$

2. Express $\log(x) + \log(y) - 2\log(z)$ as a single logarithm
   
   $\log(x) + \log(y) - \log(z) = \log(xy) - 2\log(z)$
   
   $= \log(xy) - \log(z^2)$
   
   $= \log\left(\frac{xy}{z^2}\right)$

3. Evaluate $\log_64 + \log_654 + \log_7(1)$
   
   $\log_64 + \log_654 + \log_7(1) = \log_64 + \log_654 + 0$
   
   $= \log_64 + \log_654$
   
   $= \log_6(4 \times 54)$
   
   $= \log_6(216)$
   
   $= 3$

Let’s work through some more examples involving the logarithm laws.

The first example requires us to evaluate the logarithm of 15 with base 15 plus the logarithm of 3 with base 3. From the first logarithm law we know that both of these logarithms are simply equal to 1, and hence the sum of the logarithms is equal to 2.

The second example requires us to express the logarithm of $x$, plus the logarithm of $y$, minus two times the logarithm of $z$, as a single logarithm. We can simplify the first two logarithms in this expression using the third logarithm law, to give the logarithm of $x$ times $y$ minus two times the logarithm of $z$. We can then simplify the second logarithm in our new expression using the fifth logarithm law, to give the logarithm of $x$ times $y$ minus the logarithm of $z$ squared, and finally we can simplify these two logarithms using the fourth logarithm law to give the logarithm of $x$ times $y$ divided by $z$ squared.

The third example requires us to evaluate the logarithm of 4 with base 6, plus the logarithm of 54 with base 6, plus the logarithm of 1 with base 7. The last of these logarithms can be simplified easily using the second logarithm law to give zero, and therefore our expression becomes the logarithm of 4 with base 6 plus the logarithm of 54 with base 6. We can then use the third logarithm law to simplify this to the logarithm of 4 times 54 with base 6, which is equal to the logarithm of 216 with base 6. Finally, evaluating this logarithm gives a solution of 3.
Activity 3: Practice Questions

Click on the Activity 3 link in the right-hand part of this screen.

Now have a go at using the logarithm laws on your own by completing some practice questions.

Exponentials and Logarithms

Now that we have covered the logarithm laws, we can look at how to derive the relationship between exponentials and logarithms. This is as follows:

\[ a = b^c \]

\[ \therefore \log_b(a) = \log_b(b^c) \]

\[ \therefore \log_b(a) = c \log_b(b) \quad \text{(by Logarithm Law 5)} \]

\[ \therefore \log_b(a) = c(1) \quad \text{(by Logarithm Law 1)} \]

\[ \therefore \log_b(a) = c \]

\[ \therefore c = \log_b(a) \]

Now that we have covered the logarithm laws, we can look at how to derive the relationship between exponentials and logarithms; namely, how to convert between an exponential equation in terms of positive numbers \( a \) and \( b \), and integer \( c \) to a logarithmic equation in terms of these same variables.

For the first step of working, consider that if any two values are equal then the logarithms of them, provided they have the same base, are also equal. Hence we can take the logarithm with base \( b \) of both sides, and we can then simplify the right hand side of this new equation using the fifth logarithm law. Next, we can us the first logarithm law to simplify the logarithm of \( b \) with base \( b \) to 1, and hence the right hand side of our equation simply becomes \( c \). Finally, we can simply switch the sides to leave us with an equation that sets \( c \) equal to the logarithm of \( a \) with base \( b \), as required.
End of Topic

Congratulations, you have completed this topic.
You should now have a better understanding of Logarithms.